Invited Lecture

Beyond Procedural Skills: Affordances of Typical Problem for the Teaching of Mathematics

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ABSTRACT In this paper, I will focus on how teachers can use typical problems to develop both conceptual fluency as well as procedural skills. These typical problems will be considered as mathematical tasks that teachers can opt to select, reformulate or create new ones for use in class. Teachers may find it easier to make sense of the mathematics and pedagogical considerations in the implementation of these tasks, as they are readily available in textbooks and past examination papers as compared to rich tasks. I will demonstrate that typical problems do have affordances for developing conceptual fluency and in that sense are equally good, if not better, better than the so-called rich tasks. With limited time at their disposal, teachers have to be strategic in noticing the affordances of typical problems and in optimally using their available time for selecting and using relevant tasks for their lessons. As such, the gist of the paper, connecting to my previous work, will be on how teachers can use typical problems in their day-to-day practice to enhance the learning of their students, against a backdrop of teacher noticing.

Keywords: Typical problems; Rich tasks; Affordances; Teacher noticing.

1. Introduction

1.1. Mathematical tasks

The teaching of mathematics is no doubt a complex activity. In his model for pedagogical reasoning and action Shulman (1987) stated that prior to instruction, there are two important stages that the teacher sequentially goes through. First, the teacher has to comprehend the content related to the topic to be taught and second, he or she needs to transform that comprehension to delineate individual concepts and skills and to package these into suitable tasks for students to demonstrate their learning. Mathematical tasks are central to students' learning because "tasks convey messages about what mathematics is and what doing mathematics entails" (National Council of Teachers of Mathematics [NCTM], 1991, p. 24). Accordingly, one of the daily concerns of mathematics teachers is whether to use existing tasks, create new tasks or reformulate existing tasks for use in their day-to-day practice.

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The teachers' role is to set appropriate mathematical tasks to elicit certain anticipated learning outcomes. Accordingly, Henningsen and Stein (1997) claimed that the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged. So, what constitutes a mathematical task may depend on the perspectives of both the teacher and the student. The teacher has to select the task and the student has to agree to work on the task. Stein, Grover and Henningsen (1996) described a task as a "classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). The mathematical idea could be a particular concept which the teacher wishes to teach to the students. On the other hand, Watson and Thompson (2015) refer to a task as the written presentation of a planned mathematical experience for a learner, which could be one action or a sequence of actions that form an overall experience. As such, a task could consist of a single problem or a sequence of problems, a textbook exercise, or even a sequence of more complex problems that may be interdisciplinary in nature.

Lester (1983) considers mathematical problems as tasks. He describes a problem as a task for which: (1) the individual or group confronting it wants or needs to find a solution; (2) there is not a readily accessible procedure that guarantees or completely determines the solution; and (3) the individual or group must make an attempt to find a solution. He further adds that, "Posing the cleverest problems is not productive if students are not interested or willing to attempt to solve them." (p. 232) An important point comes to the fore: a task on its own may not make much sense. Of prime importance is the willingness of the individual or group to work on the task.

1.2. Embeddedness of tasks-lessons-units

It is to be noted that in any unit of study in mathematics, there is a sequence of lessons used by a teacher and within each lesson, the teacher uses a sequence of tasks. Sometimes the teacher uses a single task in the lesson but commonly uses more than one task in a specified sequence in the lesson, as shown in Fig. 1 (see Choy and Dindyal, 2021). It is through mathematical tasks that mathematics teachers "transform" (see Shulman 1987) what they comprehend about the subject into appropriate chunks for eliciting expected learning outcomes from the students.

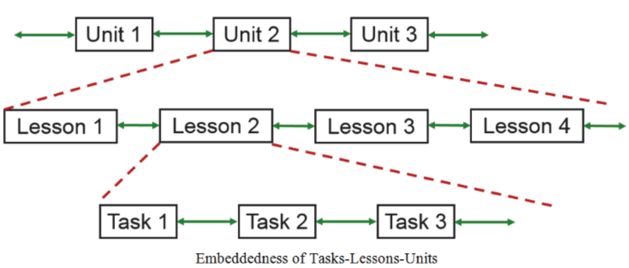

Fig. 1. The embeddedness of tasks, lessons, and units

1.3. Mathematical task as a classroom activity

Fig. 2 illustrates sequentially how a mathematical task is represented in instructional materials, how it is set up by the teacher in the classroom, and how it is implemented by students in the classroom leading to students' learning outcomes (see Henningsen and Stein, 1997).

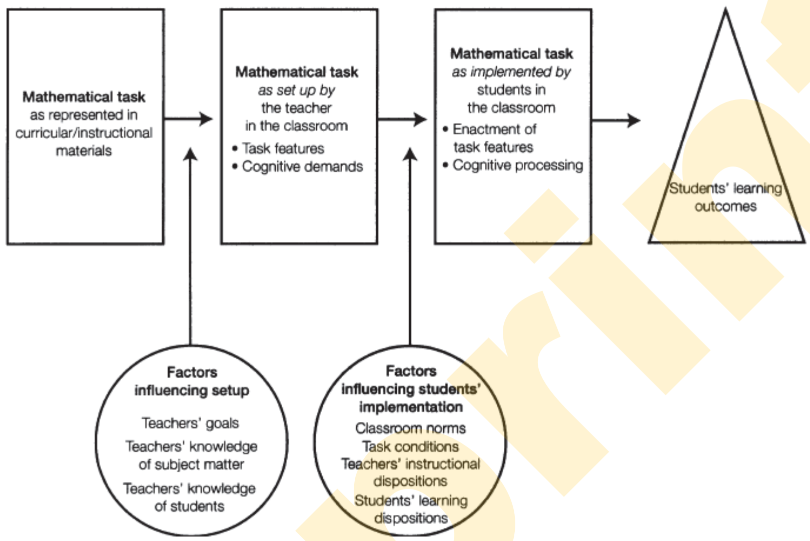

Fig. 2. Mathematical task as a classroom activity (from Henningsen and Stein, 1997)

2. Types of Tasks

2.1. Rich task or high-level task

Consider the following Task A (see Dindyal, 2018):

> *Task A* Unit cubes are used to make larger cubes of other sizes. The surface area of each of the large cubes are painted and then disassembled into the original unit cubes. For each large cube, investigate how many of the unit cubes are painted on three faces, two faces, one face, and no faces? Describe the patterns you observe.

Without a manipulative, this task involves the students visualising the larger cubes and systematically recording their observations to find useful patterns. Students may approach the task by using simpler cases or what Mason et al. (2010) called specialising. Considering $2 \times 2 \times 2$, $3 \times 3 \times 3$, $4 \times 4 \times 4$ and $5 \times 5 \times 5$ cubes, and generalising to the $n \times n \times n$ cube, several patterns can be observed. There are eight unit cubes with paint on three faces; $12(n-2)$ cubes with paint on two faces; $6(n-2)^2$ unit cubes with paint on one face; and $(n-2)^3$ unit cubes with paint on no faces. We note that the patterns generate constant, linear, quadratic and cubic functions respectively. The teacher can explore this task in various ways with the students by providing some guiding questions: For a $3 \times 3 \times 3$ cube, which unit cubes will be painted on three faces? Which unit cubes will be painted on only two faces? Which unit cubes will be painted on only one face? Which unit cubes will not be painted at all? It is to be noted that not every student will benefit from these scaffolding questions. Some questions that the teacher may additionally ask at the implementation stage: can the number of cubes with paint on one face be 1800 for any large cube? How many unit cubes will you need to make a large cube that will have 729 cubes with no paint on any face?

Task A can be considered as a rich mathematical task (see Grootenboer, 2009). It has those task features that mathematics educators have identified as important considerations for the development of mathematical understanding, reasoning and sense making (see Henningsen and Stein, 1997). Also, the cognitive demand of Task A is high. Similar tasks have been used quite extensively in the literature but termed differently. Amongst others, we have worthwhile mathematical tasks (NCTM, 1991), challenging tasks (Sullivan et al., 2014), high-level tasks (Henningsen and Stein, 1997), and open-ended tasks (Zaslavsky, 1995). While acknowledging the benefits of using such tasks, research has also surfaced some shortcomings. These high-level tasks are often more complex and take longer for implementation and may even evolve into less demanding forms of cognitive activity (see Henningsen and Stein, 1997). Such tasks are generally not meant for developing procedural skills but rather to enhance conceptual understanding. Choy and Dindyal (2017) have added that despite the affordances of challenging tasks in enhancing learning experiences, there are at least three obstacles that hinder the prevalent use of these tasks in the classrooms: (1) These tasks may be too difficult for many students, and so additional prompts or supports are needed (Sullivan et al., 2014), (2) It is time-consuming for teachers to select, adapt, or design challenging tasks to use, and (3) The inherent complexity of the tasks would involve mathematics from across the curriculum and such tasks are best implemented across several lessons, or after a few topics are taught. Henningsen and Stein (1997) have also cautioned that the mere presence of high-level mathematical tasks in the classroom will not automatically result in students' engagement in doing mathematics.

2.2. Typical problem

Consider the following Task B (see Dindyal, 2019)

> *Task B* Mary puts six identical red balls and four identical blue balls in a bag. She then takes out two balls at random from the bag, without replacement. Find as a fraction in its simplest form that she draws one red ball and one blue ball.

This task is based on elementary concepts of probability and as a problem can be solved by applying a standard procedure. A student working on this task may or may not use a tree diagram. Similar problems are quite common in examination papers and in standard textbooks. The focus here is not so much on solving the task but much more on the fact that this task is what can be termed as a typical problem, as it has a fairly straightforward answer with a very moderate difficulty level, that students have practised previously.

Typical problems are used by teachers very often on a day-to-day basis. Choy and Dindyal (2017) describe typical problems as:

…standard examination-type questions or textbook-type questions which focus largely on developing procedural fluency and at times, conceptual understanding. These questions can be solved more quickly than challenging tasks and are used frequently in mathematics lessons. Given the omnipresence of such questions in textbooks and other curriculum materials, we see typical problems as an untapped resource that can be used to orchestrate daily learning experiences. Using tasks developed from typical problems to orchestrate learning experiences would position mathematical learning experiences as an integral part of mathematics lessons, and not just reserved for occasional "enrichment" lessons. (p. 158)

Typical problems need not always be contextual. A non-contextual problem such as the one shown below can also be considered as a typical problem.

Solve the quadratic equation, $x^2 - 4x - 5 = 0$,

Typical problems can be solved in a procedural manner where the focus of the teacher might be on the development of certain specific skills. To summarise, we can say that typical problems

1. are standard examination-type questions or textbook-type questions, contextual or non-contextual, which focus largely on developing procedural fluency;
2. can be solved by students in less time;
3. are omnipresent as compared to "rich tasks" or "challenging tasks";
4. are easier for teachers to access, modify, adapt and use in class as compared to rich tasks and so are used more often in class as compared to rich tasks.

Are there ways in which teachers can engage students in mathematical tasks for developing their procedural skills as well as their conceptual fluency while using simple day-to-day tasks? The type of tasks that are readily available to teachers are typical problems, which include regular textbook problems and examination-type questions. Do typical problems have affordances (see Gibson, 1986) for developing conceptual fluency besides the expected development of procedural skills?

3. Affordances

3.1. Theory of affordances and typical problems

Henningsen and Stein (1997) have highlighted that even high-level tasks or challenging tasks have affordances to be implemented routinely by teachers. On the other hand, a legitimate question to ask, is "to what extent do typical problems, usually used for developing procedural skills have affordances for developing conceptual fluency?". What does the term affordances mean in this context of using mathematical tasks?

The perceptual psychologist, Gibson (1986) coined the term “affordances” in his famous book, *The Theory of Affordances: An Ecological Approach to Visual Perception*. Gibson claimed that, “The affordances of the environment are what it offers the animal, what it provides or furnishes, either for good or ill.” (p. 127) Gibson came up with three important ideas about affordances: (1) an affordance of an object exists in relation to an observer; (2) the affordance of the object does not change as the need of the observer changes; and (3) the observer may or may not perceive or attend to the affordance according to his needs, but the affordance being invariant is always there to be perceived. He also added that an affordance is not bestowed upon an object by the need of an observer and his act of perceiving it. This idea of affordances can be applied to typical problems, if we consider the teacher to be the observer and the typical problem or task to be the object. We can state that, (1) an affordance for using a typical problem to develop conceptual fluency exists relative to the action and capabilities of the teacher; (2) the existence of the affordance is independent of the teacher’s ability to perceive it; and (3) the affordance does not change as the needs and goals of the teacher change. Thus, we can say that to perceive the affordances of a typical problem means to be able to notice the characteristics of the task in relation to the particular understandings of the related concept in order to adapt the task for use in classrooms. According to Gibson, affordances can be perceptible or hidden. In the context of typical problems, we may consider the use of typical problems in a routine way to develop procedural skills as a perceptible affordance, because that is how they are expected to be used. On the other hand, the use of typical problems in a non-routine manner to develop conceptual fluency may be considered as a hidden affordance as this type of use of typical problems is not expected. Gibson had added that affordances that exist in relation to an observer could be positive or negative which in the context of mathematical tasks may mean a more productive or less productive use of the task in class by the teacher. To realise the positive affordance of something, Gibson suggested that we need to magnify its optical structure to that degree necessary for the behavioural encounter. How do we magnify the positive affordances of typical problems? One way is to reformulate or modify the typical problems.

3.2. Alice and the reformulation of a typical problem

In an earlier paper (Choy and Dindyal, 2017), we described how Alice, an experienced teacher enhanced the positive affordances of a typical problem on matrix multiplication (her first problem in the lesson) by reformulating the problem (modifying the problem). In this class she used four typical problems. She basically reworded the original matrix problem (not shown here) for the students avoiding the use of any matrices and thus opening up the solution space.

The reformulated typical problem:

> Teresa and Robert attend the same school. They keep a record of the awards they have earned and the points gained. Teresa obtained 29 Gold, 10 Silver, and 5 Bronze awards.

> Robert obtained 30 Gold, 6 Silver, and 8 Bronze awards. They gained 5 points from each Gold award, 3 points for each Silver award, and 2 points for each Bronze award.
>
> Find the total number of points that Teresa gained.
>
> Find the total number of points that Robert gained.

As such, the students could provide both arithmetic solutions and solutions with matrix multiplication. After the students worked on the problem, Alice orchestrated a mathematically productive discussion (Smith and Stein, 2011) by carefully attending to students' answers sequentially during the whole class discussions. Alice's orchestration of instructional activities differed from the five practices in two important ways. First, Alice used a selection of four contextual tasks on matrix multiplication, taken from past-year examination papers. This stands in contrast to Smith and Stein's idea of using a single rich task for the lesson. Second, although Alice's way of orchestrating discussion seems to reflect the five practices, Alice interjected to explain the connections in between the different solutions, instead of connecting the solutions at the end of the presentation. This provided opportunities for her to emphasise the connections between matrix multiplication and arithmetic to provide meaning to matrix operations in between students' presentations (see Choy and Dindyal, 2018). Hence, Alice kept the concept in focus and ensured coherence in the discussion by co-constructing the explanations for the different approaches with her students (Lampert et al., 2010). We can also consider Alice as using mini-cycles of the five practices by Smith and Stein (anticipating, monitoring, selecting, sequencing, and connecting), one cycle for each of the four typical problems that she used in the class.

4. Modifying Typical Problems

To notice the affordances of a typical problem, a teacher should be able to modify the typical problem in various ways, just as Alice did. The modification of typical problems can surface approaches to implementation of these problems in class either for developing procedural skills or for developing conceptual fluency. To modify typical problems, I suggest the following procedure: (1) focus on the given typical problem; (2) focus on the kind of modification that suits your implementation needs; (3) focus on the implementation of the problem in your class; and (4) after the implementation in your class, review the problem and its use and decide whether to keep it or to further modify it for subsequent use. Let's further look into these ideas as detailed in Dindyal (2018).

4.1. Focusing on the given problem

The teacher needs to check, carefully the unit on which the typical problem is based and be aware of the expected learning outcomes. All skills and sub-skills for solving the problem should be considered and the use of any results, techniques, or conventions should be highlighted. Another important idea to check is the concepts on which the problem is based, and the teacher should ask the questions: Do my students have the necessary resources to solve the problem? Is the problem within the requirements of the syllabus? Additionally, the teacher should check for the kind of structure that is already present in the statement of the typical problem and specifically look at supporting diagrams, charts, graphs, tables, parts, and subparts, etc. The wording of the problem should be scrutinized for the language, in particular, the teacher should check all action verbs (find, state, calculate, etc.), key words (all, some, at least, at the most, etc.), technical/mathematical terms, connectives, etc. The teacher should solve the problem in multiple ways, if necessary. In the end, the teacher should look out for the kind of affordances that the problem provides for enhancing the students' learning experience.

4.2. Focusing on the modification of the problem

The teacher needs to bear in mind the kinds of skills that he or she wants to elicit from the students and ask some questions such as: Do they have to draw something? Do they have to calculate, solve or prove? One idea to consider is whether to make the problem harder or simpler. The teacher should think about reducing or increasing the structure in the problem by drawing or removing diagrams, adding or reducing the number of parts and sub-parts, using or not using a table, graph or chart. The teacher should decide whether to keep the same action verb and key words as in the original problem or to select new ones. Strategies to modify the problem include: changing the numbers in the problem, changing the context of the problem, changing what is given and what is to be found, creating an open-ended problem, or integrating with another topic in mathematics. It is also worthwhile to focus on the procedural skills and the conceptual knowledge that would be elicited through working on this problem in the lesson. The use of the problem should be considered both as a stand-alone problem or embedded in a sequence of other problems by the teacher.

4.3. Focusing on the implementation of the problem

The implementation of the problem in the classroom is another aspect that should be considered carefully by the teacher. When and how will the problem be used? How will the lesson be orchestrated using this problem? Anticipate all possible solutions from the students to the given problem and which ones would be highlighted for the whole class. Using the problem in the classroom involves continual assessment throughout the lesson. One idea to focus one is: What kind of questions to ask to deepen the students' understanding? During the implementation process, the teacher may consider which other problems and in which sequence, could he or she use to enhance the students' learning.

4.4. Reflecting on the use and modification of the problem

No problem used in the classroom is perfect. There is always room for improving either the statement of the problem itself or its implementation. Accordingly, it is important to review the use of a typical problem either on its own or in conjunction with other problems. Questions to ask might include: Was this problem used optimally to enhance my students' learning? What other affordances does the problem provide that can be explored further? So, what kind of skills or dispositions should teachers possess to be able to reformulate and use typical problems?

5. Teacher Noticing

5.1. Teacher noticing and typical problems

Whether teachers select, modify or create tasks for use in class, they have to see and make sense of the mathematics and pedagogical considerations in the implementation of these tasks. Mason (2002) has previously stated that, "Every act of teaching depends on noticing: noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next" (p. 7). Ball (2011) has cautioned that although noticing is a natural part of sense making, the kind of noticing required in teaching is not a natural extension of being observant in everyday life. We can consider mathematics teacher noticing as the process of attending to students' mathematical ideas and making sense of the information to make decisions in an instructional context (Jacobs, Lamb, and Philipp, 2011; van Es and Sherin, 2008). This specialised seeing, sense making, and decision making is a set of three inter-related skills referred to by researchers as teacher noticing (Mason, 2002; Sherin, Jacobs and Philipp, 2011). More specifically the three processes are attending, interpreting and responding. For a more detailed focus on the conceptualisations of noticing see Dindyal et al. (2021).

Referring to Alice (Choy and Dindyal, 2017), we note that Alice noticed the different solutions of her students: the arithmetic solution, the solution from two separate matrix products, and the solution from a single matrix product. Alice interpreted the solutions of her students based on her experience as a teacher and in line with the learning about matrix multiplication. She interpreted each solution on its merit; she did not say that the answers were correct or incorrect; she interpreted which solutions were most important for later discussion; and she interpreted which sequence about the solutions would be most meaningful for discussions. Regarding her responding, Alice decided to send a student who had an arithmetic solution to write it on the board first. She then asked for the student who had two separate matrix products to write his solution on the board. And finally, she asked the student who had a single matrix product to write his solution on the board. Perhaps, the most important part in the discussion was when she asked if there was a different way to represent the solution as a single matrix.

Referring to Fig. 3, we can say that besides noticing the mathematics embedded in the typical problems (teacher-task-mathematics), Alice also noticed her students' thinking as they worked on the task (students-task-mathematics) and harnessed their answers as she orchestrated a whole-class discussions (teacher-students-mathematics) based on her interpretation of students' thinking as captured in their answers to the task (teacher-task-students). Connecting Alice's noticing and the affordances of the task, we note that Alice used shorter cycles of Smith and Stein's (2011) five practices—Anticipating, Monitoring, Selecting, Sequencing, and Connecting; one for each of the four problems she used in the observed lesson. She attended to students' possible confusion about using matrices to represent information in different ways (different order of matrices and matrix multiplication) and she recognised the affordances of a typical task and used it to lead students gain new insights (both conceptual and procedural fluency).

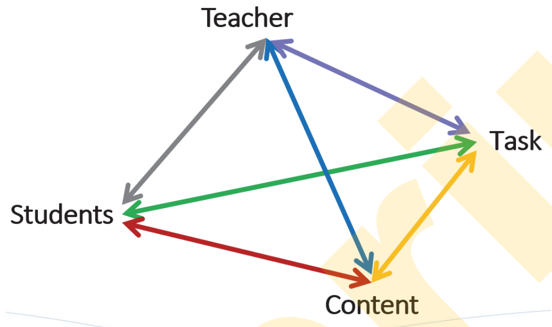

Fig. 3. Socio-didactical tetrahedron for using tasks (adapted from Rezat and Sträßer, 2012)

6. Conclusion

Henningsen and Stein (1997) have claimed that the nature of tasks can potentially influence and structure the way students think and can serve to limit or to broaden their views of the subject matter with which they are engaged. Do the tasks have to be the so-called "rich tasks" or "high-level tasks"? I do see the value in using such tasks, as they do have a place in the curriculum. Such tasks have a high cognitive demand because they test Higher Order Thinking (HOT); make connections between different concepts and topic areas across the curriculum and are best implemented across several lessons or after a few topics have been covered. However, the methods of solution for such problems are not obvious and are generally harder for students, for which some students need prompts and cues. Besides such problems are quite difficult for teachers to select, adapt and design for classroom practice. Also, they are generally, used for developing conceptual fluency and not developing procedural skills.

On the other hand, typical problems are standard examination-type questions or textbook-type questions which focus largely on developing procedural fluency and at times, conceptual understanding; they can be solved by students in less time; they are omnipresent as compared to rich tasks; can be used for developing procedural skills, but they can also be used for developing conceptual fluency; and they are easier for teachers to access, reformulate or modify, adapt and use in class (Choy and Dindyal, 2017; Choy and Dindyal, 2018). It is to be noted that teachers can reproduce several such problems from their own example space. These problems can be used individually or in a planned sequence and are more user friendly to students as compared to rich tasks. Furthermore, typical problems can be used to develop students' reasoning (Dindyal, 2019).

The important role of teachers in using typical problems should not be underestimated. Henningsen and Stein (1997) have stated that, "Not only must the teacher select and appropriately set up worthwhile mathematical tasks, but the teacher must also proactively and consistently support students' cognitive activity without reducing the complexity and cognitive demands of the task." (p. 546) When using typical problems, mathematics teachers progressively increase the complexity and the cognitive demand of the task. It is clear that a teacher's own resources (knowledge and skills) which can largely be characterised by not only the teacher's content knowledge (CK) but also by his or her pedagogical content knowledge (PCK) as well as his or her pedagogical reasoning (see Shulman, 1987), are crucial elements in the teachers' ability to confidently use typical problems in the class. With limited time at their disposal, teachers have to be strategic in optimally using time for selecting and using relevant tasks for their lessons. As such, my point is that the use of typical problems has to be examined against a backdrop of teacher noticing that can be useful to teachers in their day-to-day practice. As humans, we are all involved in acts of noticing and more so in acts of not noticing. We can confidently say, "so do teachers". Some teachers can reformulate and use typical problems in pedagogically more meaningful ways while others cannot. Schoenfeld (2011) has cautioned that noticing is consequential "… what you see or don't see shapes what you do or don't do." (p. 228)

There is enough evidence to suggest that just as rich tasks or high-level tasks can be used in cognitively less demanding ways, typical problems do offer affordances to be implemented in cognitively more demanding ways. It is difficult for teachers to create new rich tasks or high-level tasks or even modify them for their specific use. On the other hand, typical problems are easily available, can be reformulated easily and provide a fairly smooth entry point for most students, even the lower performing students. It is worthwhile for teachers to consider the use of typical problems beyond the development of procedural skills and explore the affordances to develop conceptual fluency.

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